**Technical Appendix to**
**"The Carbon Tax: Welfare Triangle, or Welfare Obelisk?"**
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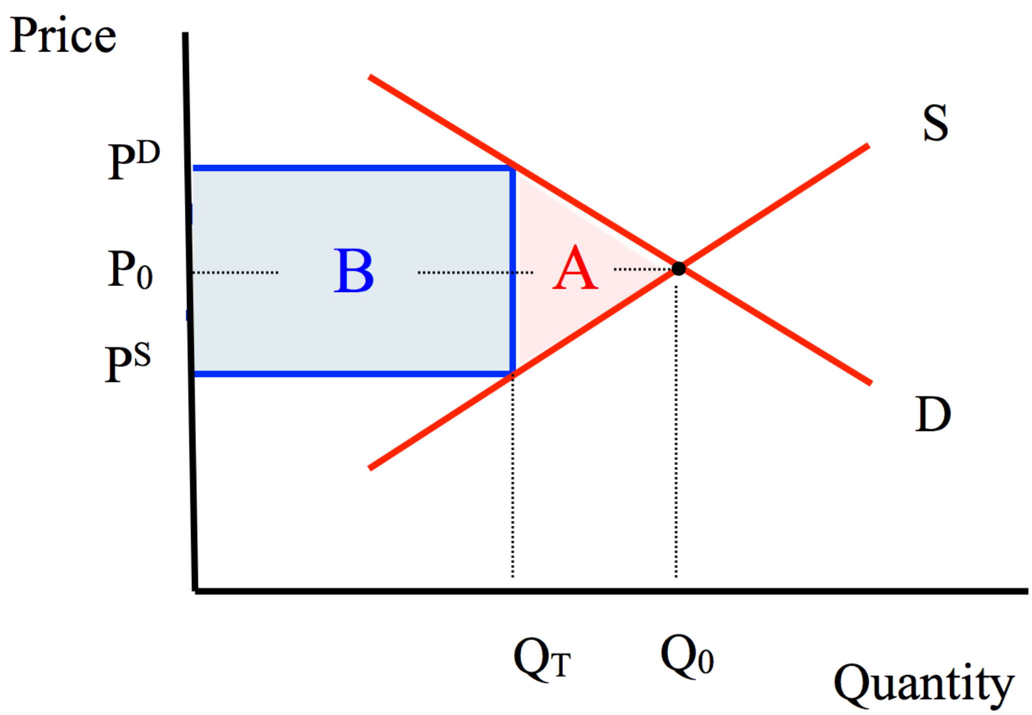


**The Welfare Triangle**

The textbook analysis of the taxation of commodities whose production or consumption impose "negative externalities," e.g. environmental costs on outsiders, begins with the "welfare triangle," illustrated as Area A in the figure above. In this figure, the demand curve D indicates the quantity Q of the commodity in question that consumers are willing to purchase at each price P, while the supply curve S indicates the quantity producers are willing to produce at each price. In the absence of a tax, the market tends to clear at the price $P_0$ and quantity $Q_0$ at which the supply and demand curves intersect.

If a tax T equal to the height of triangle A and rectangle B is imposed on the commodity, the quantity produced and sold falls to $Q_T$. The demand price paid by consumers rises to $P^D$, while the supply price received by producers falls to $P^S$, where $P^D - P^S = T$. This outcome is the same whether consumers or producers formally pay the tax to the government.

Areas A and B together measure the combined loss in "Consumers' Surplus" and "Producers' Surplus" caused by the tax (Willig 1976). However, area B represents the tax revenues to the government. If we may assume that the government spends this money on services that are equal in value to the expenditure, or uses it to efficiently reduce other distortionary taxes, the net "Deadweight Welfare Loss" to the economy as a whole is triangle A by itself

With straight line supply and demand curves, the quantity produced Q(T) will equal

$$Q(T) = Q_0 - bT,$$

for some positive constant *b* that depends on the slopes of the supply and demand curves. The area of the welfare triangle A is then

$$WC(T) = \text{area}(A) = (b/2)T^2.$$

If the commodity imposes an environmental cost

$$EC(T) = cQ(T),$$

where *c* is a positive constant (in the case of a carbon tax, the SSC), then it can easily be shown that total cost TC(T), i.e. the combined welfare cost and environmental cost,

$$\begin{aligned} TC(T) &= WC(T) + EC(T) \\ &= (b/2)T^2 + cQ(T) \\ &= (b/2)T^2 + c(Q_0 - bT), \end{aligned}$$

is minimized when T = c. This is the implicit assumption of Taylor and of the authors of the Whitehouse-Schatz bill.

**Or is it a Welfare Obelisk?**

When the welfare cost of taxation is represented by the "welfare triangle" A, any negative environmental externality, no matter how small, justifies at least a small environmental "Pigovian" tax (named for economist A.C. Pigou). This is because the welfare cost is proportional to the *square* of the tax, and therefore is "of the second order of smalls", while the environmental cost of the output is, at least for a small tax, directly proportional to the output, and therefore is "of the first order of smalls."

However, if government expenditures are completely wasteful (which admittedly does sometimes seem to be the case), the welfare cost of a tax is the much larger area of the obelisk-shaped region A + B, which greatly alters the elementary Pigovian prescription. In fact, if the government is completely inefficient, which seems to be the tacit assumption of Murphy et al., there is no middle ground between a zero tax for moderate externalities and closing the industry down entirely for severe externalities!

Under the pessimistic assumption that government expenditures are completely wasteful, the welfare cost of a tax T is

$$\begin{aligned} WC(T) &= \text{area}(A) + \text{area}(B) \\ &= (b/2)T^2 + T(Q_0 - bT). \end{aligned}$$

The total cost then becomes

$$TC(T) = WC(T) + EC(T)$$
$$= (b/2)T^2 + T(Q_0 - bT) + c(Q_0 - bT).$$

This cost is of the first order of smalls near $T = 0$, so that it no longer follows that a Pigovian tax is always justified. In fact, with straight line supple and demand schedules, this cost is quadratic in T, but with a *negative second derivative*, so that the first order conditions that indicate a cost minimum using the welfare triangle now indicate a *cost maximum*. There therefore can only be a "corner solution" to the cost minimization problem, either at $T = 0$ or at $T \geq T_0$, where

$$T_0 = Q_0 / b$$

is the tax that completely shuts down the industry.

It can be shown that if $c < T_0/2$, total cost is minimized with $T = 0$, whereas if $c > T_0/2$, total cost is minimized with a prohibitive tax $T \geq T_0$. Since even the Administration's perhaps too generous estimates of the SCC do not come close to half of what it would take to close down the fossil fuel industry, the "welfare obelisk" argument would support Murphy et al.'s recommendation of $T = 0$.

However, even though most government operations are at least somewhat inefficient, if only because of the ubiquitous "principal-agent problem," the government surely has some legitimate functions that it should be funding despite this inefficiency. Let *w* be the average wastefulness of government spending, where *w* lies somewhere between 0 (perfect efficiency) and 1 (perfect wastefulness). Then the total fiscal and environmental cost of a tax T is

$$TC(T) = \text{area}(A) + w\ \text{area}(B) + EC(T).$$

Although this cost is less than with the full welfare obelisk, it can still be shown that it necessarily leads to a corner solution unless $w < 0.5$, i.e. unless government spending is at least 50% efficient. If $w > 0.5$, it is still true that the Murphy et al. solution $T = 0$ is optimal so long as $c < T_0/2$, which appears to be the case for carbon emissions, even using the highest Administration estimates of the SCC.

Assuming (generously) that $w < 0.5$, the second derivative of cost is again positive as in the Pigovian case $w = 0$, so that an interior solution to the cost minimization problem obtains. In this case, the optimal environmental tax takes the form

$$T = f\ c,$$

for some *f* between 0 and 1. It can easily be shown (simple calculus-based Econ problem!) that

$$f = (1 - wT_0/c)/(1 - 2w)$$

if $c > w\ T_0$, and 0 otherwise. If $c < w\ T_0$, the first-order welfare cost of taxation exceeds the first-order environmental gain from reducing Q, and no tax is justified. However, there is a substantial range in which *f* is greater than 0 but less than 1.

In summary, if government spending is inefficient (but less than 50% inefficient), a carbon tax equal to some fraction of the SCC may be justified. Any carbon tax should therefore not be simply set equal to the estimated SCC as assumed by Taylor and the authors of the Whitehouse-Schatz bill.

**Or is it a ... Welfare Trapezoid?**

Another valid point that Murphy et al. raise against the elementary Pigovian analysis is that it assumes that the market in question starts off with no distortions. However, if there is initially a universal revenue tax, every market will already have a triangular welfare burden that is increased by a *trapezoidal region* (not illustrated) when a further environmental tax is imposed on a particular sector. Since the area of this trapezoid increases in the first order of smalls when the environmental tax is added to the revenue tax, the case for an environmental tax is again weakened. In fact, the optimal environmental tax will again take the form *f c*, where *f* again lies somewhere between 0 and 1.

Suppose there are two outputs in the economy, $Q_1$ and $Q_2$, and that a total tax $T_1$ is placed on the first output and $T_2$ on the second output. Total revenue is then

$$R = T_1Q_1 + T_2Q_2.$$

Assume for simplicity that both industries are the same size and have the same shaped straight-line demand and supply curves, so that

$$Q_{i\_} = Q_0 - bT_i,\ i = 1, 2.$$

(It is assumed here for simplicity that each market is not affected by the price or quantity in the other market. A general equilibrium analysis that takes these interactions into account might affect the results somewhat.)

Abstracting from the inefficiency of government spending (i.e. assuming w = 0 in the previous section), the welfare cost in each sector is measured by its welfare triangle. If $Q_1$ also imposes an environmental cost $cQ_1$ on the economy, total cost is

$$TC(T_1, T_2) = (b/2)T_1^2 + (b/2)T_2^2 + cQ_1.$$

If $c = 0$, efficient taxation requires $T_1 = T_2$, with the level of the common tax rate determined by the revenue target R. However, with $c > 0$, the optimal $T_1$ will generally exceed $T_2$, again by some fraction *f* of *c* as in the preceding section.

Since there are now two unknowns, $T_1$ and $T_2$, the math is a little more complicated, and requires the use of a Lagrange Multiplier on the revenue constraint. (Good advanced Econ question!) In brief, it can readily be shown that the first order conditions imply

$$T_1 = T_2 + f\,c,$$

where

$$f = 1 - 2\,T_2/T_0.$$

Since with straight line demand and supply curves, revenue is maximized when the tax equals $T_0/2$, it makes no sense for the purely revenue tax $T_2$ to be greater than $T_0/2$, so that *f* will indeed lie in the range (0, 1) so long as there is a background revenue tax on the economy as a whole, even when government spending is perfectly efficient. The level of $T_1$ and $T_2$ is then determined by the revenue target R.

Again, it is difficult to say what value of *f* is implied by the US tax system, but it is surely no more than 0.5. Taking both inefficiencies into account (i.e. the Welfare Obelisk in addition to the Welfare Trapezoid) would in fact result in an even lower value of *f* than 0.5. Setting f = 0.5 as in the proposal is therefore a very generous carbon tax.

(I gather from the Wikipedia article, "Pigovian Tax," that the analysis of the welfare cost of taxation has advanced far beyond Pigou's original "welfare triangle," so that this brief note may to some extent be re-inventing the wheel.)

**References**

Robert D. Willig,"Consumer's Surplus without Apology," *American Economic Review* **66** (Sept. 1976): 589-97.

Wikipedia article, "Pigovian Tax," https://en.wikipedia.org/wiki/Pigovian_tax.